TWO-DIMENSIONAL MIXED BOUNDARY VALUE PROBLEM IN HEAT AND MASS TRANSFER FOR A CHARACTERISTIC EQUATION WITH MULTIPLE ROOTS

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A system of two differential equations of parabolic type is examined. A boundary value problem is set up and solved. A system of integrodifferential equations is obtained for determining the unknown functions. A method of reducing this to a system of ordinary Volterra integral equations is given.

Consider the system of differential equations

$$\frac{\partial U_i}{\partial t} = \sum_{k=1}^{2} a_{ik}\left(\frac{\partial^2 U_k}{\partial x^2} + \frac{\partial^2 U_k}{\partial y^2}\right) \quad (i=1,2). \tag{1}$$

We impose the following conditions on the real coefficients a_{ik}:

$$(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) = 0, \quad a_{11}+a_{22} > 0. \tag{2}$$

If (2) is satisfied, the roots of the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \tag{3}$$

will be positive and multiple, i.e., $\lambda_1 = \lambda_2 = \lambda > 0$.

With these assumptions we shall solve the following boundary problem.

Problem: To find a solution of the system of equations (1) in the region $P[t > 0,\ 0 < x < l,\ -\infty < y < +\infty]$, satisfying the initial condition

$$U_i(x, y, t)|_{t=0} = 0 \quad (i = 1,2) \tag{4}$$

and the boundary conditions

$$\begin{gathered}(\alpha_1^{(1)}U_1 + \alpha_2^{(1)}U_2)\Big|_{x=0} = \psi_1(y, t), \quad \left(\frac{\partial U_1}{\partial x} + h_1 U_1\right)\Big|_{x=0} = \psi_2(y, t), \\ (\alpha_1^{(2)}U_1 + \alpha_2^{(2)}U_2)\Big|_{x=l} = \psi_3(y, t), \quad \left(\frac{\partial U_2}{\partial x} + h_2 U_2\right)\Big|_{x=l} = \psi_4(y, t),\end{gathered} \tag{5}$$

where α_i^k, h_i (i, k = 1, 2) are given constants; $\psi_i(y, t)$ (i = 1, 2, 3, 4) are known continuous finite functions having continuous finite partial derivatives of sufficiently high order, and $\psi_i(y, 0) = 0$. This problem was solved in [3] for the case when the roots of the characteristic equation (2) are positive and different.

We shall seek a solution in the following form [4, 5]:

$$\begin{aligned} U_i(x, y, t) &= \sum_{k,j=1}^{2} B_{ij}^k g^{(1)} * \omega_{1k}[x, y, t] - \sum_{k=1}^{2} B_{i2}^k g^{(2)} * \omega_{1k}[x, y, t] + \\ &+ \sum_{k,j=1}^{2} B_{ij}^k g^{(1)} * \omega_{2k}[l-x, y, t] - \sum_{k=1}^{2} B_{i2}^k g^{(2)} * \omega_{2k}[l-x, y, t], \end{aligned} \tag{6}$$

where

$$g^{(1)}(x, y, t) = \frac{1}{2\pi\lambda t}\exp\left[-\frac{x^2+y^2}{4\lambda t}\right], \quad g_x^{(1)} = \frac{\partial}{\partial x} g^{(1)}, \quad g_{xx}^{(1)} = \frac{\partial^2}{\partial x^2} g^{(1)};$$

$$g^{(2)}(x, y, t) = \frac{x^2+y^2}{8\pi\lambda^2 t^2}\exp\left[-\frac{x^2+y^2}{4\lambda t}\right], \quad g_x^{(2)} = \frac{\partial}{\partial x} g^{(2)}, \quad g_{xx}^{(2)} = \frac{\partial^2}{\partial x^2} g^{(2)};$$

$$g * \omega[x, y, t] = \int_0^t d\tau \int_{-\infty}^{+\infty} g(x, y-\eta, t-\tau)\,\omega(\eta, \tau)\,d\eta.$$

The coefficients B_{ij}^k in (6) are defined as follows [4]:

$$B_{11}^1 = a_{11}, \; B_{12}^1 = \lambda - a_{11}, \; B_{11}^2 = a_{12}, \; B_{12}^2 = -a_{12},$$
$$B_{21}^1 = a_{21}, \; B_{22}^1 = -a_{21}, \; B_{21}^2 = a_{22}, \; B_{22}^2 = \lambda - a_{22}. \tag{7}$$

It is then easy to verify that the functions $U_i(x, y, t)$ given by (6) satisfy (1) and (4).

The unknown functions $\omega_{ik}(y, t)$ $(i, k = 1, 2)$ in (6) must be defined so that functions $U_i(x, y, t)$ still satisfy boundary conditions (5). For this purpose functions (6) must be substituted in (5).

We first point out the following lemmas:

Lemma 1: If the function $\omega(y, t)$ has a finite derivative $\partial\omega/\partial t$ and $\omega(y, 0) = 0$, then

$$\lim_{x\to 0}\frac{\partial^2}{\partial x^2}\int_0^t d\tau \int_{-\infty}^{\infty} \frac{\omega(\eta, \tau)[x^2+(y-\eta)^2]}{8\pi\lambda^2(t-\tau)^2}\exp\left[-\frac{x^2+(y-\eta)^2}{4\lambda(t-\tau)}\right]d\eta =$$
$$= -\frac{1}{2\lambda} g^{(1)} * \frac{\partial\omega}{\partial\tau}[0, y, t]. \tag{8}$$

Proof: We first transform the integral under the derivative sign in (8) as follows:

$$J = \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{\omega(\eta, \tau)[x^2+(y-\eta)^2]}{8\pi\lambda^2(t-\tau)^2}\exp\left[-\frac{x^2+(y-\eta)^2}{4\lambda(t-\tau)}\right]d\eta =$$
$$= -\int_{-\infty}^{+\infty} d\eta \int_0^t \frac{\omega(\eta, \tau)}{2\pi\lambda}\frac{\partial}{\partial\tau}\exp\left[-\frac{x^2+(y-\eta)^2}{4\lambda(t-\tau)}\right]d\tau. \tag{9}$$

Integrating the inner integral on the right side of (9) by parts, we obtain

$$J = \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{1}{2\pi\lambda}\frac{\partial\omega(\eta, \tau)}{\partial\tau}\exp\left[-\frac{x^2+(y-\eta)^2}{4\lambda(t-\tau)}\right]d\eta.$$

Then

$$\lim_{x\to 0}\frac{\partial^2}{\partial x^2}J = \lim_{x\to 0}\int_0^t d\tau \int_{-\infty}^{+\infty} \frac{1}{2\pi\lambda}\frac{\partial\omega(\eta, \tau)}{\partial\tau}\frac{\partial^2}{\partial x^2}\exp\left[-\frac{x^2+(y-\eta)^2}{4\lambda(t-\tau)}\right]d\eta =$$

$$= \lim_{x\to 0}\int_0^t d\tau \int_{-\infty}^{+\infty} \frac{1}{2\pi\lambda}\frac{\partial\omega(\eta, \tau)}{\partial\tau}\left[-\frac{1}{2\lambda(t-\tau)} + \frac{x^2}{4\lambda^2(t-\tau)^2}\right]\exp\left[-\frac{x^2+(y-\eta)^2}{4\lambda(t-\tau)}\right]d\eta.$$

Since

$$\lim_{x\to 0}\int_0^t d\tau \int_{-\infty}^{+\infty} \frac{\partial\omega(\eta, \tau)}{\partial\tau}\frac{x^2}{4\pi\lambda^2(t-\tau)^2}\exp\left[-\frac{x^2+(y-\eta)^2}{4\lambda(t-\tau)}\right]d\eta = 0,$$

We have

$$\lim_{x\to 0}\frac{\partial^2}{\partial x^2}J = -\frac{1}{2\lambda}\int_0^t d\tau \int_{-\infty}^{+\infty} g^{(1)}(0, y-\eta, t-\tau)\frac{\partial\omega(\eta, \tau)}{\partial\tau}d\eta =$$

$$= -\frac{1}{2\lambda} g^{(1)} * \frac{\partial \omega}{\partial \tau}[0,\ y,\ t],$$

which it was required to prove.

Lemma 2: If the function ω(y, t) has finite derivatives $\frac{\partial \omega(y,\ t)}{\partial t}$, $\frac{\partial^2 \omega(y,\ t)}{\partial y^2}$ and ω(y, 0) = 0, then

$$\lim_{x\to 0} \int_0^t d\tau \int_{-\infty}^{+\infty} g_{xx}^{(1)}(x,\ y-\eta,\ t-\tau)\,\omega(\eta,\ \tau)\,d\eta = \frac{1}{\lambda} g^{(1)} * F[\omega][0,\ y,\ t],$$

where $F[\omega] = \frac{\partial \omega}{\partial \tau} - \lambda \frac{\partial^2 \omega}{\partial \eta^2}$. The proof of this lemma may be found in [3].

We now substitute (6) into (5), using the properties of double layer "potentials" [4, 5] and Lemmas 1 and 2. We then obtain

$$\begin{aligned} -\alpha_1^{(1)}\omega_{11} - \alpha_2^{(1)}\omega_{12} + \sum_{k,\ j=1}^{2} (\alpha_1^{(1)} B_{1j}^k + \alpha_2^{(1)} B_{2j}^k)\, g_x^{(1)} * \omega_{2k}[l,\ y,\ t] - \\ - \sum_{k=1}^{2} (\alpha_1^{(1)} B_{12}^k + \alpha_2^{(1)} B_{22}^k)\, g_x^{(2)} * \omega_{2k}[l,\ y,\ t] = \psi_1(y,\ t), \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{1}{\lambda} \sum_{k,\ j=1}^{2} B_{1j}^k g^{(1)} * F[\omega_{1k}][0,\ y,\ t] + \frac{1}{2\lambda} \sum_{k=1}^{2} B_{12}^k g^{(1)} * \frac{\partial \omega_{1k}}{\partial \tau}[0,\ y,\ t] + \\ + \sum_{k,\ j=1}^{2} B_{1j}^k g_{xx}^{(1)} * \omega_{2k}[l,\ y,\ t] - \sum_{k=1}^{2} B_{12}^k g_{xx}^{(2)} * \omega_{2k}[l,\ y,\ t] - h_1\omega_{11} + \\ + h_1 \sum_{k,\ j=1}^{2} B_{1j}^k g_x^{(1)} * \omega_{2k}[l,\ y,\ t] - h_1 \sum_{k=1}^{2} B_{12}^k g_x^{(2)} * \omega_{2k}[l,\ y,\ t] = \psi_2(y,\ t), \end{aligned} \tag{11}$$

$$\begin{aligned} \alpha_1^{(2)}\omega_{21} + \alpha_2^{(2)}\omega_{22} + \sum_{k,\ j=1}^{2} (\alpha_1^{(2)} B_{1j}^k + \alpha_2^{(2)} B_{2j}^k)\, g_x^{(1)} * \omega_{1k}[l,\ y,\ t] - \\ - \sum_{k=1}^{2} (\alpha_1^{(2)} B_{12}^k + \alpha_2^{(2)} B_{22}^k)\, g_x^{(2)} * \omega_{1k}[l,\ y,\ t] = \psi_3(y,\ t), \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{1}{\lambda} \sum_{k,\ j=1}^{2} B_{2j}^k g^{(1)} * F[\omega_{2k}][0,\ y,\ t] + \frac{1}{2\lambda} \sum_{k=1}^{2} B_{22}^k g^{(1)} * \frac{\partial \omega_{2k}}{\partial \tau}[0,\ y,\ t] + \\ + \sum_{k,\ j=1}^{2} B_{2j}^k g_{xx}^{(1)} * \omega_{1k}[l,\ y,\ t] - \sum_{k=1}^{2} B_{22}^k g_{xx}^{(2)} * \omega_{1k}[l,\ y,\ t] + h_2\omega_{22} + \\ + h_2 \sum_{k,\ j=1}^{2} B_{2j}^k g_x^{(1)} * \omega_{1k}[l,\ y,\ t] - h_2 \sum_{k=1}^{2} B_{22}^k g_x^{(2)} * \omega_{1k}[l,\ y,\ t] = \psi_4(y,\ t). \end{aligned} \tag{13}$$

Thus, for determining the unknown functions $\omega_{ik}(y, t)$ $(i, k = 1, 2)$ we have a system of four integrodifferential equations (10)-(13), which we solve by reducing it to a system of four integral equations. Thus we eliminate the function $\omega_{12}(y, t)$ from (10) and (11), and $\omega_{21}(y, t)$ from (12) and (13), assuming that $\alpha_2^{(1)} \neq 0$ and $\alpha_1^{(2)} \neq 0$*. Then

$$\frac{1}{\lambda}\sum_{j=1}^{2}(\alpha_2^{(1)} B_{1j}^1 - \alpha_1^{(1)} B_{1j}^2) g^{(1)} * F[\omega_{11}][0, y, t] + \frac{1}{2\lambda}(\alpha_2^{(1)} B_{12}^1 - \alpha_1^{(1)} B_{12}^2) \times$$
$$\times g^{(1)} * \frac{\partial \omega_{11}}{\partial \tau}[0, y, t] = \alpha_2^{(1)} h_1 \omega_{11} + \varphi_2(y, t) - \sum_{k=1}^{2} H_1^k * \omega_{2k}[l, y, t], \tag{14}$$

$$\frac{1}{\lambda}\sum_{j=1}^{2}(\alpha_2^{(2)} B_{2j}^1 - \alpha_1^{(2)} B_{2j}^2) g^{(1)} * F[\omega_{22}][0, y, t] + \frac{1}{2\lambda}(\alpha_2^{(2)} B_{22}^1 - \alpha_1^{(2)} B_{22}^2) \times$$
$$\times g^{(1)} * \frac{\partial \omega_{22}}{\partial \tau}[0, y, t] = \alpha_1^{(2)} h_2 \omega_{22} + \varphi_4(y, t) - \sum_{k=1}^{2} H_2^k * \omega_{1k}[l, y, t], \tag{15}$$

$$H_i^k(l, y-\eta, t-\tau) = \frac{1}{\lambda}\sum_{\nu, j=1}^{2} B_{ij}^{i'}(\alpha_1^{(i)} B_{1\nu}^k + \alpha_2^{(i)} B_{2\nu}^k) F[g_x^{(1)}] * g^{(1)}[l, y-\eta, t-\tau] -$$
$$- \frac{1}{\lambda}\sum_{j=1}^{2} B_{ij}^{i'}(\alpha_1^{(i)} B_{12}^k + \alpha_2^{(i)} B_{22}^k) F[g_x^{(2)}] * g^{(1)}[l, y-\eta, t-\tau] +$$
$$+ \frac{1}{2\lambda} B_{i2}^{i'} \sum_{\nu=1}^{2}(\alpha_1^{(i)} B_{1\nu}^k + \alpha_2^{(i)} B_{2\nu}^k)\frac{\partial}{\partial \tau}(g_x^{(1)}) * g^{(1)}[l, y-\eta, t-\tau] -$$
$$- \frac{1}{2\lambda} B_{i2}^{i'}(\alpha_1^{(i)} B_{12}^k + \alpha_2^{(i)} B_{22}^k)\frac{\partial}{\partial \tau}(g_x^{(2)}) * g^{(1)}[l, y-\eta, t-\tau] + \tag{16}$$
$$+ (-1)^{1+i}\alpha_{i'}^{(i)}\left\{\sum_{j=1}^{2} B_{ij}^k g_{xx}^{(1)}(l, y-\eta, t-\tau) - B_{i2}^k g_{xx}^{(2)}(l, y-\eta, t-\tau) + \right.$$
$$\left. + h_i \sum_{j=1}^{2} B_{ij}^k g_x^{(1)}(l, y-\eta, t-\tau) - h_i B_{i2}^k g_x^{(2)}(l, y-\eta, t-\tau)\right\}$$
$$\left(i = 1, 2; i' = \begin{cases} 1, & \text{if } i = 2 \\ 2, & \text{if } i = 1 \end{cases}\right);$$

$$\varphi_2(y, t) = \alpha_2^{(1)} \psi_2(y, t) + \frac{1}{\lambda}\sum_{j=1}^{2} B_{1j}^2 g^{(1)} * F[\psi_1][0, y, t] +$$
$$+ \frac{1}{2\lambda} B_{12}^2 g^{(1)} * \frac{\partial \psi_1}{\partial \tau}[0, y, t];$$
$$\varphi_4(y, t) = -\alpha_1^{(2)} \psi_4(y, t) + \frac{1}{\lambda}\sum_{j=1}^{2} B_{2j}^1 g^{(1)} * F[\psi_3][0, y, t] +$$
$$+ \frac{1}{2\lambda} B_{22}^1 g^{(1)} * \frac{\partial \psi_3}{\partial \tau}[0, y, t].$$

It may easily be shown that the kernels H_1^k and H_2^k given by (16) are regular.

Now instead of (14) and (15), we shall examine their characteristic equation

$$L[\omega] = T_1 g^{(1)} * F[\omega][0, y, t] + T_2 g^{(1)} * \frac{\partial \omega}{\partial \tau}[0, y, t] = f[y, t], \tag{17}$$

*If $\alpha_1^{(2)} = \alpha_2^{(1)} = 0$, we obviously must assume that $\alpha_1^{(1)} \neq 0$ and $\alpha_2^{(2)} \neq 0$. In this case the problem is solved by analogous reasoning.

where

$$T_1 = T_{11} = \frac{1}{\lambda}\sum_{j=1}^{2}(\alpha_2^{(1)} B_{1j}^1 - \alpha_1^{(1)} B_{1j}^2);\; T_2 = T_{12} = \frac{1}{2\lambda}(\alpha_2^{(1)} B_{12}^1 - \alpha_1^{(1)} B_{12}^2)$$

for the characteristic equation in (14) and

$$T_1 = T_{21} = \frac{1}{\lambda}\sum_{j=1}^{2}(\alpha_2^{(2)} B_{2j}^1 - \alpha_1^{(2)} B_{2j}^2);\; T_2 = T_{22} = \frac{1}{2\lambda}(\alpha_2^{(2)} B_{22}^1 - \alpha_1^{(2)} B_{22}^{(2)})$$

for the characteristic equation in (15).

We assume that a Fourier-Laplace transformation may be applied to the functions $\omega(y, t)$ and $f(y, t)$. Applying this transformation to both sides of (17), we obtain

$$\frac{\sqrt{2\pi}}{\sqrt{\lambda}}\left(T_1\sqrt{p+\lambda s^2} + T_2\frac{p}{\sqrt{p+\lambda s^2}}\right)\bar{\bar{\omega}}(s, p) = \bar{\bar{f}}(s, p),$$

whence

$$\bar{\bar{\omega}}(s, p) = \frac{\sqrt{\lambda}}{\sqrt{2\pi}}\,\frac{\sqrt{p+\lambda s^2}}{(T_1+T_2)p + T_1\lambda s^2}\,\bar{\bar{f}}(s, p). \tag{18}$$

If $T_1 + T_2 \neq 0$, from (18) we have

$$\bar{\bar{\omega}}(s, p) = \frac{\sqrt{\lambda}}{\sqrt{2\pi}\,(T_1+T_2)}\,\frac{\sqrt{p+\lambda s^2}}{p + As^2}\,\bar{\bar{f}}(s, p), \tag{19}$$

where $A = \lambda T_1/(T_1 + T_2)$.

If $A \geq 0$, then, applying the inverse Laplace transformation, we obtain

$$\frac{\sqrt{\lambda}}{\sqrt{2\pi}\,(T_1+T_2)}\,\frac{\sqrt{p+\lambda s^2}}{p+As^2} \rightarrow \frac{\sqrt{\lambda}}{\sqrt{2\pi}\,(T_1+T_2)}\,\frac{\exp[-\lambda s^2 t]}{\sqrt{t}} + \frac{\sqrt{\lambda}\,(\lambda - A)}{\sqrt{2\pi}\,(T_1+T_2)}\,s^2\int_0^t \frac{\exp\{-[\lambda\tau + (t-\tau)A]s^2\}}{\sqrt{\tau}}\,d\tau. \tag{20}$$

Then, applying the inverse Fourier transformation to the right side of (20), we get

$$\frac{\sqrt{\lambda}}{\sqrt{2\pi}(T_1+T_2)}\,\frac{\sqrt{p+\lambda s^2}}{p+As^2} \stackrel{\rightarrow}{\sim} \frac{1}{\sqrt{2\pi}(T_1+T_2)t}\exp\left[-\frac{y^2}{4\lambda t}\right] + \frac{\sqrt{\lambda}\,(\lambda - A)}{2\sqrt{2\pi}\,(T_1+T_2)}\int_0^1 \frac{1}{\sqrt{z}\,a^3(z)t}\left[1 - \frac{y^2}{2a^2(z)t}\right] \times \exp\left[-\frac{y^2}{4a^2(z)t}\right]dz = \Gamma(0, y, t), \tag{21}$$

where $a(z) = \sqrt{\lambda z + (1-z)A}$ and the symbol $\stackrel{\rightarrow}{\sim}$ denotes that the inverse transform must be found first, before the inverse Fourier transformation is applied.

If $A < 0$, then

$$\frac{\sqrt{\lambda}}{\sqrt{2\pi}(T_1+T_2)}\,\frac{\sqrt{p+\lambda s^2}}{p+As^2} \rightarrow \frac{\sqrt{\lambda(\lambda - A)}}{\sqrt{2\pi}(T_1+T_2)}\,s\exp[-As^2 t] + \tag{22}$$

$$+\frac{\sqrt{\lambda}}{2\sqrt{2\pi}(T_1+T_2)}\int_t^\infty \frac{\exp\{-[\lambda\tau+(t-\tau)A]s^2\}}{\tau^{3/2}}d\tau. \quad (22) \text{ (cont'd)}$$

The first term on the right side of (22) contains the function $\exp[-As^2t]$, for which, when $A < 0$, an inverse Fourier transformation does not exist. Therefore, for the whole right side of (22), when $A < 0$, an inverse Fourier transformation likewise does not exist.

Generalizing the results obtained, we may formulate the solvability theorem:

Theorem: If $T_1/(T_1 + T_2) < 0$, then Eq. (17) does not have a solution, and if $T_1/(T_1 + T_2) \geq 0$, (17) is solvable and the function

$$\omega(y,\ t)=L^{-1}[f]=\int_0^t d\tau\int_{-\infty}^{+\infty}\Gamma(0,\ y-\eta,\ t-\tau)f(\eta,\ \tau)d\eta=\Gamma * f[0,y,t] \quad (23)$$

is a solution, if $f(y, t)$ has continuous finite derivatives $\partial f/\partial t$, $\partial^2 f/\partial y^2$. Substituting (23) into (17), we in fact verify that the function $\omega(y, t)$ given by (23) satisfies (17). We may therefore remove all the conditions previously imposed on $\omega(y, t)$ and $f(y, t)$ for application of the Fourier-Laplace transformation.

For the characteristic equations in (14) and (15), the condition of solvability is formulated as follows.

If $A_1=T_{11}/(T_{11}+T_{12})>0$, i.e., if $\alpha_2^{(1)}>0$ and $\alpha_2^{(1)}(a_{11}+3a_{22})+2\alpha_1^{(1)}a_{12}>0$, or $\alpha_2^{(1)}<0$ and $\alpha_2^{(1)}(a_{11}+3a_{22})+2\alpha_1^{(1)}a_{12}<0$, then the characteristic equation in (14) is solvable.

If $A_2=T_{21}/(T_{21}+T_{22})>0$, i.e., if $\alpha_1^{(2)}>0$ and $\alpha_1^{(2)}(3a_{11}+a_{22})+2\alpha_2^{(2)}a_{21}>0$, or $\alpha_1^{(2)}<0$ and $\alpha_1^{(2)}(3a_{11}+a_{22})+2\alpha_2^{(2)}a_{21}<0$, then the characteristic equation in (15) is solvable.

We shall call the expression $\Gamma(0,\ y,\ t)$ the resolvent of (17), and $\Gamma_1(0,\ y,\ t)$ and $\Gamma_2(0,\ y,\ t)$ the resolvents of (14) and (15).

We assume that the characteristic equations from (14) and (15) satisfy the conditions of solvability. Then, applying the inverse operator L^{-1} to (14) and (15), we obtain

$$\omega_{11}(y,\ t)=\alpha_2^{(1)}h_1\Gamma_1*\omega_{11}[0,y,t]-\sum_{k=1}^{2}H_1^k*\Gamma_1*\omega_{2k}[l,y,t]+\Gamma_1*\varphi_2[0,y,t], \quad (24)$$

$$\omega_{22}(y,\ t)=\alpha_1^{(2)}h_2\Gamma_2*\omega_{22}[0,y,t]-\sum_{k=1}^{2}H_2^k*\Gamma_2*\omega_{1k}[l,y,t]+\Gamma_2*\varphi_4[0,y,t]. \quad (25)$$

We substitute the values of ω_{11} and ω_{22} found from (24) and (25) into (10) and (12). We then obtain new integral equations, which form a system of four integral equations with regular kernels:

$$\omega_{1j}(y,\ t)=\int_0^t d\tau\int_{-\infty}^{+\infty}K_{1j}^{(1)}(0,y-\eta,\ t-\tau)\omega_{11}(\eta,\ \tau)d\eta+$$
$$+\sum_{k=1}^{2}\int_0^t d\tau\int_{-\infty}^{+\infty}K_{1j}^{(2)}(l,\ y-\eta,\ t-\tau)\omega_{2k}(\eta,\tau)d\eta+F_{1j}(y,\ t), \quad (26)$$

$$\omega_{2j}(y,\ t)=\int_0^t d\tau\int_{-\infty}^{+\infty}K_{2j}^{(1)}(0,\ y-\eta,\ t-\tau)\omega_{22}(\eta,\tau)d\eta+$$

$$+\sum_{k=1}^{2}\int_0^t d\tau\int_{-\infty}^{+\infty}K_{2j}^{(2)}(l,\ y-\eta,\ t-\tau)\omega_{1k}(\eta,\ \tau)d\eta+F_{2j}(y,\ t), \quad (27)$$

where

$$K_{11}^{(1)} = \alpha_2^{(1)} h_1 \Gamma_1(0, y-\eta, t-\tau); \; K_{12}^{(1)} = -\alpha_1^{(1)} h_1 \Gamma_1(0, y-\eta, t-\tau);$$

$$K_{11}^{(2)} = -H_1^k * \Gamma_1[l, y-\eta, t-\tau];$$

$$K_{12}^{(2)} = \frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} H_1^k * \Gamma_1[l, y-\eta, t-\tau] + \frac{1}{\alpha_2^{(1)}} \sum_{j=1}^{2} (\alpha_1^{(1)} B_{1j}^k + \alpha_2^{(1)} B_{2j}^k) \times$$

$$\times g_x^{(1)}[l, y-\eta, t-\tau] - \frac{1}{\alpha_2^{(1)}} (\alpha_1^{(1)} B_{12}^k + \alpha_2^{(1)} B_{22}^k) g_x^{(2)}(l, y-\eta, t-\tau);$$

$$K_{22}^{(1)} = \alpha_1^{(2)} h_2 \Gamma_2(0, y-\eta, t-\tau); \; K_{21}^{(1)} = -\alpha_2^{(2)} h_2 \Gamma_2(0, y-\eta, t-\tau);$$

$$K_{22}^{(2)} = -H_2^k * \Gamma_2[l, y-\eta, t-\tau];$$

$$K_{21}^{(2)} = \frac{\alpha_2^{(2)}}{\alpha_1^{(2)}} H_2^k * \Gamma_2[l, y-\eta, t-\tau] - \frac{1}{\alpha_1^{(2)}} \sum_{j=1}^{2} (\alpha_1^{(2)} B_{1j}^k + \alpha_2^{(2)} B_{2j}^k) \times$$

$$\times g_x^{(1)}(l, y-\eta, t-\tau) + \frac{1}{\alpha_1^{(2)}} (\alpha_1^{(2)} B_{12}^k + \alpha_2^{(2)} B_{22}^k) g_x^{(2)}(l, y-\eta, t-\tau);$$

$$F_{11} = \Gamma_1 * \varphi_2[0, y, t]; \; F_{12} = -\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} \Gamma_1 * \varphi_2[0, y, t] - \frac{1}{\alpha_2^{(1)}} \psi_1(y, t);$$

$$F_{22} = \Gamma_2 * \varphi_4[0, y, t]; \; F_{21} = -\frac{\alpha_2^{(2)}}{\alpha_1^{(2)}} \Gamma_2 * \varphi_4[0, y, t] + \frac{1}{\alpha_1^{(2)}} \psi_3(y, t).$$

We may obtain the following estimates by direct calculation:

$$|F_{ij}(y, t)| \leqslant M, \; |K_{ij}^{(\nu)}| \leqslant \frac{P}{t-\tau} \exp\left[-\delta^2 \frac{(y-\eta)^2}{t-\tau}\right], \tag{28}$$

where M, P, and δ are some positive constants.

We shall seek a solution of system (26), (27) by the method of successive approximations in the form of the series

$$\omega_{ij}(y, t) = \omega_{ij}^{(0)}(y, t) + \omega_{ij}^{(1)}(y, t) + \ldots + \omega_{ij}^{(n)}(y, t) + \ldots, \tag{29}$$

where

$$\omega_{ij}^{(0)}(y, t) = F_{ij}(y, t);$$

$$\omega_{1j}^{(n)}(y, t) = \int_0^t d\tau \int_{-\infty}^{+\infty} K_{1j}^{(1)}(0, y-\eta, t-\tau) \omega_{11}^{(n-1)}(\eta, \tau) d\eta +$$

$$+ \sum_{k=1}^{2} \int_0^t d\tau \int_{-\infty}^{+\infty} K_{1j}^{(2)}(l, y-\eta, t-\tau) \omega_{2k}^{(n-1)}(\eta, \tau) d\eta;$$

$$\omega_{2j}^{(n)}(y, t) = \int_0^t d\tau \int_{-\infty}^{+\infty} K_{2j}^{(1)}(0, y-\eta, t-\tau) \omega_{22}^{(n-1)}(\eta, \tau) d\eta +$$

$$+ \sum_{k=1}^{2} \int_0^t d\tau \int_{-\infty}^{+\infty} K_{2j}^{(2)}(l, y-\eta, t-\tau) \omega_{1k}^{(n-1)}(\eta, \tau) d\eta.$$

Using (28), it is easy to show that

$$|\omega_{ij}^{(n)}(y, t)| \leqslant \frac{2MP_1^n}{\Gamma[(n+2)/2]} t^{\frac{n}{2}}, \text{ where } P_1 = 3P\,\Gamma\left(\frac{1}{2}\right) \Big/ \delta.$$

Since $\Gamma\left(\frac{n+2}{2}\right) \Big/ \Gamma\left(\frac{n+3}{2}\right) \to 0$ as $n \to \infty$, series (29) converges absolutely and uniformly for $0 \leq t \leq T$.

The results obtained may be generalized as follows: our mixed boundary value problem (1)-(5) has a solution when, and only when the characteristic equations from (14) and (15) are simultaneously solvable, which is possible if the coefficients a_{ik}, $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ satisfy one of the following four conditions:

$$\begin{cases} \alpha_2^{(1)} > 0 \\ \alpha_2^{(1)}(a_{11}+3a_{22}) + 2\alpha_1^{(1)}a_{12} > 0 \\ \alpha_1^{(2)} > 0 \\ \alpha_1^{(2)}(3a_{11}+a_{22}) + 2\alpha_2^{(2)}a_{21} > 0 \end{cases} \qquad \begin{cases} \alpha_2^{(1)} > 0 \\ \alpha_2^{(1)}(a_{11}+3a_{22}) + 2\alpha_1^{(1)}a_{12} > 0 \\ \alpha_1^{(2)} < 0 \\ \alpha_1^{(2)}(3a_{11}+a_{22}) + 2\alpha_2^{(2)}a_{21} < 0 \end{cases}$$

$$\begin{cases} \alpha_2^{(1)} < 0 \\ \alpha_2^{(1)}(a_{11}+3a_{22}) + 2\alpha_1^{(1)}a_{12} < 0 \\ \alpha_1^{(2)} > 0 \\ \alpha_1^{(2)}(3a_{11}+a_{22}) + 2\alpha_2^{(2)}a_{21} > 0 \end{cases} \qquad \begin{cases} \alpha_2^{(1)} < 0 \\ \alpha_2^{(1)}(a_{11}+3a_{22}) + 2\alpha_1^{(1)}a_{12} < 0 \\ \alpha_1^{(2)} < 0 \\ \alpha_1^{(2)}(3a_{11}+a_{22}) + 2\alpha_2^{(2)}a_{21} < 0. \end{cases}$$

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